MATH 1010E 9/8/2014

Last time: • $f(x) = \frac{9x+1}{x^2-3x+2} = c$ need to consider $c=0$ separately.

• $\exp(x)$, $\ln(x)$ defined as infinite series.

More about $\exp(x)$

Recall: $$e := \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n.$$

$$\exp(x) := \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

Theorem: $\boxed{\exp(x) = e^x}$

"Pf": Step 1: $\exp(1) = e$

i.e. $$\sum_{k=0}^{\infty} \frac{1}{k!} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n.$$

By binomial thm.

$$\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k}$$

$$= \sum_{k=0}^{n} \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} \frac{1}{n^k}$$

$$= \sum_{k=0}^{n} \frac{1}{k!} \underbrace{\left(1-\frac{1}{n}\right)}_{\substack{\downarrow \\ 1}} \underbrace{\left(1-\frac{2}{n}\right)}_{\substack{\downarrow \\ 1}} \cdots \underbrace{\left(1-\frac{k-1}{n}\right)}_{\substack{\downarrow \\ 1}} \quad \text{as } n\to\infty.$$

$$\to \sum_{k=0}^{\infty} \frac{1}{k!} \quad \text{as } n\to\infty$$

$$\| $$

$$\exp(1).$$

~~Step 2~~:

<u>Step 2</u> : $\exp(x) = e^x$.

$$e^x = \left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right]^x = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{nx}$$

$$= \lim_{n\to\infty}\left(1+\frac{x}{nx}\right)^{nx} \overset{m=nx}{=} \lim_{m\to\infty}\left(1+\frac{x}{m}\right)^m$$

$$= \lim_{m\to\infty}\sum_{k=0}^{m}\binom{m}{k}\frac{x^k}{m^k} = \sum_{k=0}^{\infty}\frac{x^k}{k!} = \exp(x).$$

\#.

More Properties of $\exp(x) = e^x$

- $e^{x+y} = e^x \cdot e^y$
- $e^x > 0 \quad \forall x \in \mathbb{R}$

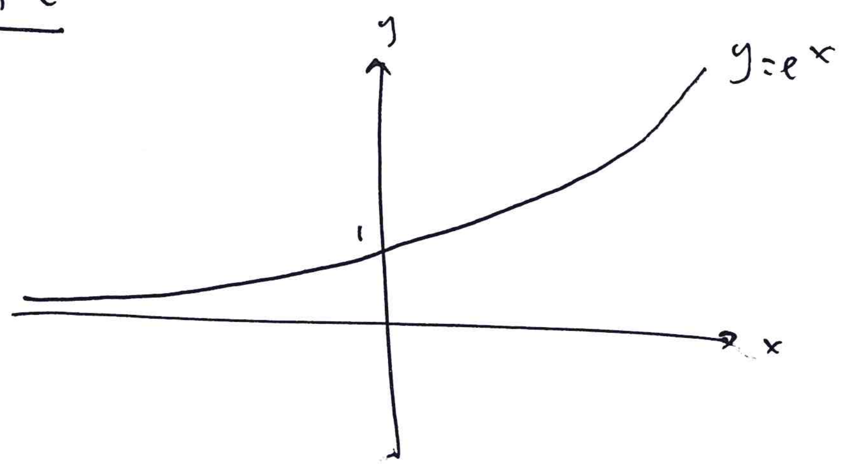
  and $e^x > 1$ for any $x > 0$
- $\exp(x)$ is an increasing function. <u>Pf:</u> $e^{x+h} = e^x \cdot \underbrace{e^h}_{>1} > e^x$. ($h>0$)

  ie $x > y \Rightarrow e^x > e^y$.
- $\lim_{x\to+\infty} e^x = +\infty$ and $\lim_{x\to-\infty} e^x = 0$. <u>Pf:</u> $e^x = 1 + x + \frac{x^2}{2} + \dots$

  $> 1 + x \to +\infty$

  as $x \to \infty$.

<u>Graph of e^x</u>



Trigonometric functions

Define $\sin : \mathbb{R} \to \mathbb{R}$, $\cos : \mathbb{R} \to \mathbb{R}$ by

$$\sin x := x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$$\cos x := 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

Fact: This agrees with the usual $\sin x$, $\cos x$ defined in trigonometry.

Ex: $\tan x := \dfrac{\sin x}{\cos x}$. Find a series definition for $\tan x$.

Sol^{n}: Let $\tan x = a_0 + a_1 x + a_2 x^2 + \cdots\cdots$

By def: $\tan x \cdot \cos x = \sin x$

$$(a_0 + a_1 x + a_2 x^2 + \cdots)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$a_0 + a_1 x + \left(a_2 - \frac{a_0}{2}\right)x^2 + \left(a_3 - \frac{a_1}{2}\right)x^3 + \cdots = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

compare coefficients

$a_0 = 0$

$a_1 = 1$

$a_2 - \dfrac{a_0}{2} = 0 \implies a_2 = 0.$

$a_3 - \dfrac{a_1}{2} = -\dfrac{1}{6} \implies a_3 = \dfrac{1}{3}.$

$$\tan x = x + \frac{x^3}{3} + \cdots\cdots$$

$\#$

Properties of trigonometric functions

(i) Periodic: $\sin(x+2\pi) = \sin x$

$\cos(x+2\pi) = \cos x$.

(ii) $\sin^2 x + \cos^2 x = 1$

(iii) Double angle Formula: $\sin 2x = 2\sin x \cos x$

$\cos 2x = \cos^2 x - \sin^2 x$.

(iv) Half angle Formula: $\sin^2 \frac{x}{2} = \frac{1-\cos x}{2}$

$\cos^2 \frac{x}{2} = \frac{1+\cos x}{2}$

Sec/csc/cot: $\sec x := \frac{1}{\cos x}$, $\csc x := \frac{1}{\sin x}$, $\cot x := \frac{1}{\tan x}$.

Ex: Prove the following identities:

(i) $1+\tan^2 x = \sec^2 x$

(ii) $1+\cot^2 x = \csc^2 x$

Q: Prove that

(i) $\sin(x \pm y) = \sin x \cos y \pm \sin y \cos x$

(ii) $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$.

Pf: later!

Hyperbolic functions

$$\sinh x := \frac{e^x - e^{-x}}{2}, \quad \cosh x := \frac{e^x + e^{-x}}{2}, \quad \tanh x := \frac{\sinh x}{\cosh x}.$$

$$\operatorname{csch} x := \frac{1}{\sinh x}, \quad \operatorname{sech} x := \frac{1}{\cosh x}, \quad \coth x := \frac{1}{\tanh x}.$$

Q: Find the series expansion of sinh / cosh.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \text{odd part of } e^x$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \text{even part of } e^x.$$

$$\boxed{e^x = \underbrace{\cosh x}_{\text{even}} + \underbrace{\sinh x}_{\text{odd}}}$$

Identities:

(1) $\cosh^2 x - \sinh^2 x = 1$

(2) $\begin{cases} 1 - \tanh^2 x = \operatorname{sech}^2 x \\ \coth^2 x - 1 = \operatorname{csch}^2 x \end{cases}$

(3) $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y.$

Pf: (3) R.H.S. $= \frac{1}{4}(e^x - e^{-x})(e^y + e^{-y}) + \frac{1}{4}(e^x + e^{-x})(e^y - e^{-y})$

$$= \frac{1}{4}\Big(e^{x+y} + \cancel{e^{x-y}} - \cancel{e^{-x+y}} - e^{-x-y} + e^{x+y} - \cancel{e^{x-y}} + \cancel{e^{-x+y}} - e^{-x-y}\Big)$$

$$= \frac{1}{2}\left(e^{x+y} - e^{-(x+y)}\right) = \sinh(x+y)$$

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Ex: Is $\cancel{\sinh x}$ $\cosh x$ 1-1 or onto ?

Sol$^{\underline{n}}$: Consider the eq$^{\underline{n}}$: $\cancel{\sinh}$ $\cosh x = c$

ie $$\frac{e^x + e^{-x}}{2} = c$$

$$\Rightarrow \quad e^x - 2c + e^{-x} = 0$$

$$\Rightarrow \quad e^{2x} - 2ce^x + 1 = 0$$

$$e^x = \frac{2c \pm \sqrt{4c^2 - 4}}{2}$$

$$x = \ln\left(c \pm \sqrt{c^2 - 1}\right)$$

<u>not onto</u> : no sol$^{\underline{n}}$ when $c^2 < 1$

<u>not 1-1</u> : 2 sol$^{\underline{n}}$ when $c^2 > 1$.

Q: Sketch the graphs of $\cosh x$, $\sinh x$, $\tanh x$?

Euler's Formula

Assume there is some "number" i st $i^2 = -1$.

$$e^{ix} = 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \cdots$$

$$= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} + \cdots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$

$$= \cos x + i \sin x$$

$$\Rightarrow \quad \cos x = \frac{e^{ix} + e^{-ix}}{2} \quad ; \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

Last time:

- $\sin x$, $\cos x$, $\tan x$
- $\sinh x$, $\cosh x$, $\tanh x$.
- $e^x = \exp$.

Limits (Ch.2 Textbook).

Idea: " $\lim_{x \to a} f(x) = L$ " $\iff$ " If x gets closer and closer to a then $f(x)$ gets closer and closer to L "

E.g. $f(x) = x^2$ $\quad f : \mathbb{R} \to \mathbb{R}$

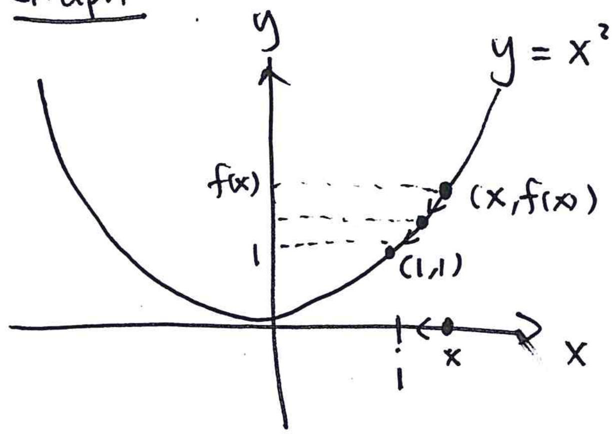
$$\boxed{\lim_{x \to 1} f(x) = ?}$$

Numerically

x	$f(x)$
1.1	1.21
1.01	1.0201
1.001	1.002001
1.0001	1.00020001
⋮	⋮
≈	≈
1	1

$$\boxed{\lim_{x \to 1} x^2 = 1}$$

Graph



$$\boxed{\lim_{x \to 1} x^2 = 1}$$

Mathematical Def^{n} (not required). ε-δ def.

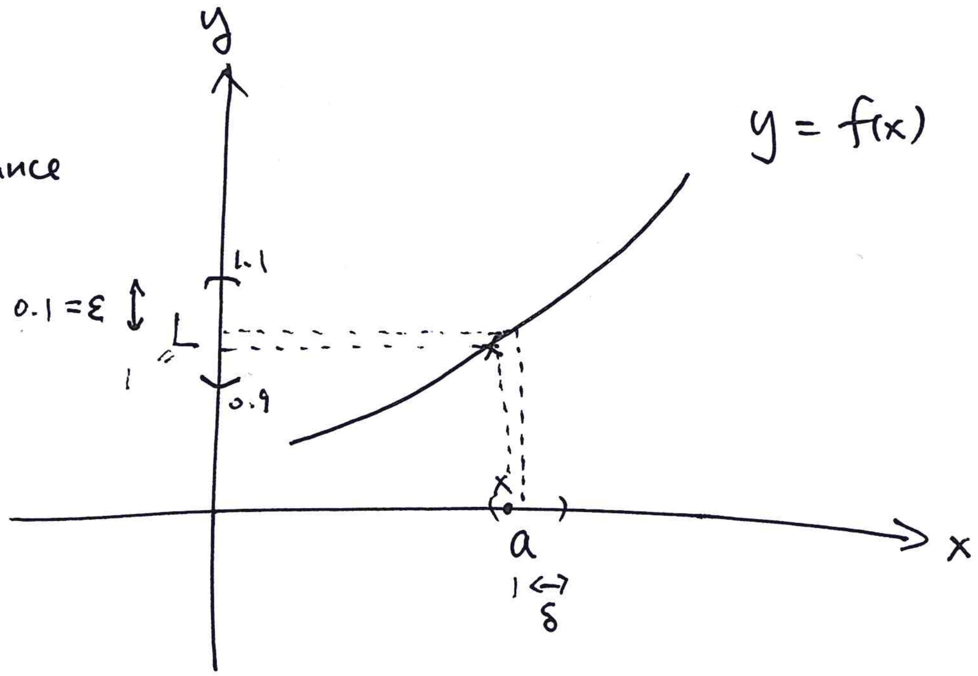
" $\lim_{x \to a} f(x) = L$ " $\iff$ " $\forall \varepsilon > 0$, $\exists\, \delta = \delta(\varepsilon) > 0$ s.t.

$|f(x) - L| < \varepsilon$ for all x s.t.

$0 < |x-a| < \delta$ "

ε : error tolerance

" Mathematical analysis "



Q: How to calculate limits? $\lim_{x \to a} f(x)$.

Rule 1 : Substitute $x=a$ into $f(x)$, if everything makes sense.

E.g. : (1) $\lim_{x \to 1} (2x^2+3x-1) = 2\cdot 1^2+3\cdot 1-1 = 2+3-1 = 4$.

(2) $\lim_{x \to \frac{\pi}{2}} (\sin x) = \sin\frac{\pi}{2} = 1$

(3) $\lim_{x \to -1} \frac{x^2-1}{x-1} = \frac{(-1)^2-1}{(-1)-1} = \frac{0}{-2} = 0$

(4)* $\lim_{x \to 1} \frac{x^2-1}{x-1} = \frac{1^2-1}{1-1} = \frac{0}{0}$ × does not make sense.

Rule 2 : Simplify expression first, then substitute.

(4) $\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}} = \lim_{x\to 1} x+1 = 1+1 = 2.$

(5) $\lim_{x\to 0} \frac{\tan x}{\sin x} = \lim_{x\to 0} \frac{\cancel{\sin x}/\cos x}{\cancel{\sin x}} = \lim_{x\to 0} \frac{1}{\cos x} = \frac{1}{1} = 1$

(6) $\lim_{x\to 4} \frac{2-\sqrt{x}}{4-x} = \lim_{x\to 4} \frac{(2-\sqrt{x})(2+\sqrt{x})}{(4-x)(2+\sqrt{x})} = \lim_{x\to 4} \frac{\cancel{4-x}}{\cancel{(4-x)}(2+\sqrt{x})}$

Recall: $(a-b)(a+b) = a^2-b^2$

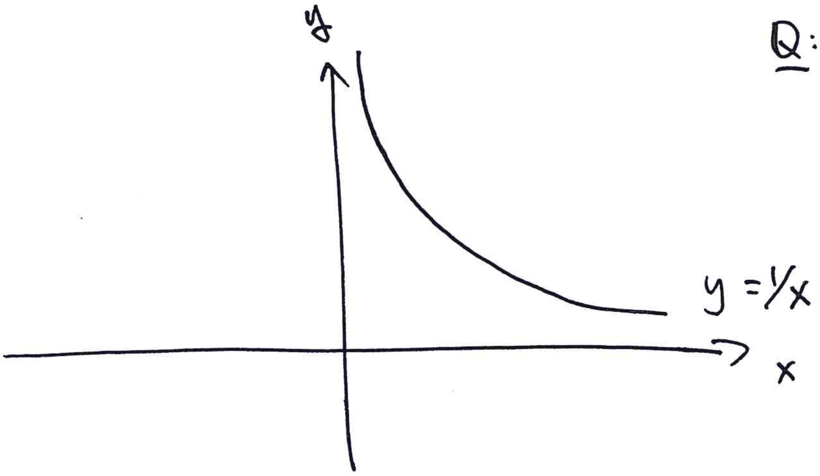
$= \lim_{x\to 4} \frac{1}{2+\sqrt{x}}$

$= \frac{1}{2+2} = \frac{1}{4}$ ※

Note 1 : To consider $\lim_{x\to a} f(x)$, the function need not be defined at $x=a$.

Note 2 : We can consider $x\to\pm\infty$ and $L=\pm\infty$.

Eg. $\lim_{\substack{x\to 0 \\ x>0}} \frac{1}{x} = +\infty$ $\qquad \lim_{x\to+\infty} \frac{1}{x} = 0$

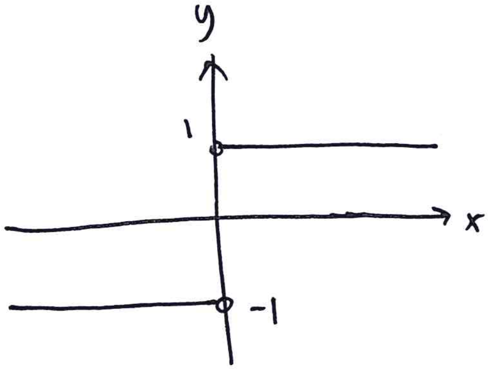


Q: $\lim_{\substack{x\to 0 \\ x<0}} \frac{1}{x} = ?$

$\lim_{x\to-\infty} \frac{1}{x} = ?$

Note 3 : Limit may not exists .

$$f(x) = \operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x>0 \\ -1 & \text{if } x<0 \end{cases}$$



$$\lim_{\substack{x\to 0\\ x>0}} f(x) = 1 \neq \lim_{\substack{x\to 0\\ x<0}} f(x) = -1$$

$$\Rightarrow \lim_{x\to 0} f(x) \text{ does not exists.}$$

Fact: (Uniqueness of Limit)

If $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} f(x) = L_2$

then $L_1 = L_2$.

A useful way to find limit

$$\text{If} \left. \begin{array}{c} \lim_{\substack{x\to a^+\\ (x>a)}} f(x) \overset{①}{=} L \\ \| ② \\ \lim_{\substack{x\to a^-\\ (x<a)}} f(x) \overset{②}{=} L \end{array} \right\} \Rightarrow \lim_{x\to a} f(x) = L .$$